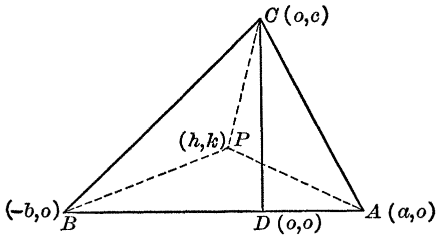
GEOMETRY.

441. Proposed by H. E. TREFETHEN, Colby College.

In the triangle ABC find the locus of points at which the sides AB and AC subtend equal angles.

I. Solution by C. N. Schmall, New York City.

Take the base AB and the altitude CD as the axes of coördinates. Then we have to find the locus of a point P such that $\angle APB = \angle APC$. Let the coördinates be A: $(a, 0)$, B: $(-b, 0)$, C: $(0, c)$. The slope of the line joining (x_1, y_1) and (x_2, y_2) is $(y_2 - y_1)/(x_2 - x_1)$. Hence the slope m_1 of PB is $k/(h+b)$; the slope m_2 of PA is $k/(h-a)$; and the slope m_3 of PC is $(k-c)/h$. Therefore, $\angle APB = \tan^{-1}[(m_1 - m_2)/(1 + m_1 m_2)]$, which, on substituting the values of m_1 and m_2, becomes



$$\tan^{-1}\frac{k(h-a) - k(h+b)}{(h-a)(h+b) + k^2}. \tag{1}$$

Likewise,

$$\angle APC = \tan^{-1}\frac{m_2 - m_3}{1 + m_2 m_3} = \tan^{-1}\frac{hk - (k-c)(h-a)}{h(h-a) + k(k-c)}. \tag{2}$$

Hence from (1) and (2), writing x, y for h, k, we have the required locus of P:

$$\frac{y(x-a) - y(x+b)}{(x-a)(x+b) + y^2} = \frac{xy - (y-c)(x-a)}{x(x-a) + y(y-c)}.$$

Upon further reduction, this becomes

$$(2a+b)y^3 + c(x - b - 2a)y^2 + (2ax + bx + ab)y + c(x-a)^2(x+b) = 0,$$

an equation of the third degree.

Also solved by V. M. Spunar.

II. Solution by Nathan Altshiller, University of Washington, Seattle.

The problem may be generalized: Find the locus of the point M such that the two given segments AB, $A'B'$ subtend at M equal positive angles (MA, MB), (MA', MB').

If the fixed positive angle (MA, MB) turns about the point M, its sides MA, MB generate two projective pencils, whose double elements are the isotropic lines MC, MC' (C, C' designate the cyclical points) passing through M. From

this projectivity results the involution $M(AA', BB', CC')$. Hence the problem takes the following projective form: Given three couples of points AA', BB', CC', find the locus of the point M such that the three couples of rays $M(AA', BB', CC')$, shall be in involution. Cayley proved analytically[1] that the locus of M is a general cubic passing through the given points. The following is an outline of a synthetic proof of this theorem and provides a method for the construction of the locus.

The four points A, A', B, B' determine a complete quadrangle (Q_1) whose diagonal triangle is PQR $[P \equiv AA', BB'; Q \equiv AB', A'B; R \equiv AB, A'B']$. Let (Q_2) be the complete quadrangle in which C, C' are two vertices and Q, R two diagonal points, S denoting the third diagonal point. The two quadrangles (Q_1), (Q_2) determine two pencils of conics (π) and (σ), which determine the same involution (I) on the line QR, the points Q and R being the double elements of (I). Let π_n and σ_n be those conics of the two pencils which determine the same couple N, N' of (I). Besides N, N' the two conics have two other points M, M' in common. M and M' belong to the required locus. The line MM' passes through the fixed point L which is the point of intersection of the two rays that correspond to the ray QR in the two involutions $Q(AA', CC')$ and $R(AA', CC')$. The point $T_n \equiv (MM', QR)$ is the harmonic conjugate of the fixed point $T \equiv (PS, QR)$ with respect to the couple NN'. The pencil of rays (L) and the pencil of conics (σ) are thus projective and the locus of the points of intersection M, M' of homologous elements of these two forms is a cubic (Chasles' theorem).

The points of the given couples may be real or conjugate imaginaries.

If the points of one of the couples, say A and A', coincide, this point will be a double point on the cubic. Such is the case in the proposed question. The required locus is therefore a circular cubic (C_3) with a double point at A (a strophoid) passing through the vertices B and C of the given triangle.

The cubic (C_3) is the pedal curve of the point A with regard to the parabola (P), which is tangent: (1) to the line BC; (2) to the bisectors C, C' of the angle BAC and its adjacent angle; (3) to the perpendiculars n, n' erected at B and C to AB and AC respectively.

Proof.—Let O, O' be the centers of two arcs of circles at all points of which the two segments AB, AC subtend equal angles. Besides A the two circles meet in a point M which belongs to (C_3). P and P' being the points diametrically opposite A in the two circles, the line PP', according to a known property of intersecting circles, passes through M and is perpendicular to AM. The points O, O' being on the perpendicular bisectors of the segments AB, AC, the points P, P' are on the perpendiculars n, n'. The angles BAO and CAO' are equal; therefore the lines AP and AP' form a couple of conjugate elements in the equilateral involution of rays at A, the double elements of which are the bisectors C, C'. Hence

$$(P \cdots) \barwedge A(P \cdots) \barwedge A(P' \cdots) \barwedge (P' \cdots).$$

[1] A. Cayley, *Collected Papers*, I, p. 184.

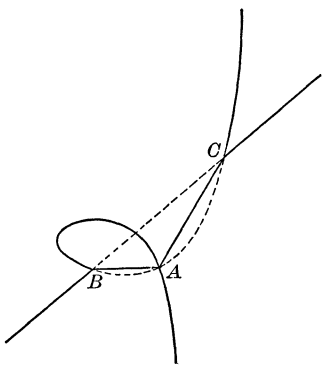
The tangents to (C_3) at A are the bisectors C, C'. The foot of the perpendicular from A upon BC belongs to (C_3).

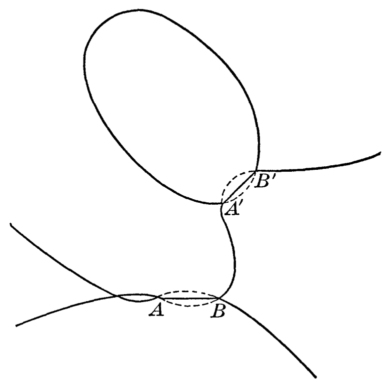
Remarks, I. In order that all the points of (C_3) shall belong to the required locus the condition of "equal angles" must be replaced by "equal positive angles." Otherwise only those parts of (C_3) that are included in the angle BAC and in its vertical angle, will posses the required property.

II. If $AC = AB$, the cubic (C_3) degenerates into the bisector of the angle BAC and the circumference of the circle circumscribed about the triangle ABC.

III. The line of centers OO' envelops a parabola similar to the parabola (P), the point A being the center of similitude.

Editorial Note.—The formulas of analytic geometry and the procedure of synthetic geometry are both such that continuous deformations do not destroy their validity. The word "subtend," having physical connotations, is not so flexible.





The locus in general, for the usual meaning of subtend, is made up of parts of two cubic curves forming a continuous curve in the projective plane without a continuous derivative. In the special case one cubic degenerates and the real part is the line BC. The heavy curves in the accompanying cuts sufficiently show the locus.

443. Proposed by C. N. SCHMALL, New York City.

A quadrilateral of any shape whatever is divided by a transversal into two quadrilaterals. The diagonals of the original figure and those of the two resulting (smaller) figures are then drawn. Show that their three points of intersection are collinear.

SOLUTION BY LAENAS G. WELD, Pullman, Ills.

[The word "(smaller)" should be omitted, as it destroys the generality of the proposition.]

Using trilinear coördinates, let ABC be the triangle of reference and designate the points in which the lines $l\alpha + m\beta + n\gamma = 0$ and $p\alpha + q\beta + r\gamma = 0$ cut the sides a, b, c by L, M, N and P, Q, R, respectively. Then $BCMN$, $BCQR$ and $QMNR$ are the three quadrilaterals in question. The equations of the diagonals are readily written as follows: